Solutions of Assignment 5

Q1. For any $i \in I$, as f_i are family of convex functions defined on $\mathcal{X} \to [-\infty, +\infty]$, so we have for any $x, y \in \mathcal{X}$, and $\lambda \in (0,1)$, we have

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$$

Put $f := \sup_{i \in I} f_i$, then we have $f_i \leq f$ for all $i \in I$. So, by fixing $i \in I$ but arbitrary, then we have

$$f_i\left(\lambda x + (1-\lambda)y\right) \leq \lambda f_i(x) + (1-\lambda) f_i(y) \leq \lambda f(x) + (1-\lambda) f(y)$$

So, we have

$$\lambda f(x) + (1-\lambda) f(y) \geq \sup_{i \in I} f_i(\lambda x + (1-\lambda)y) = f\left(\lambda x + (1-\lambda)y\right)$$

Thus, this proves that $f = \sup_{i \in I} f_i$ is convex.

Q2. (a) Since $f_1, \ldots, f_k$ are convex functions, so for any $x, y \in \bigcap_{i=1}^{k} \operatorname{dom}(f_i)$ and $\lambda \in (0,1)$, we have

$$f_i\left(\lambda x + (1-\lambda)y\right) \leq \lambda f_i(x) + (1-\lambda) f_i(y), \quad \forall i \in \{1, \ldots, k\}$$

Multiplying $w_i \geq 0$ on both sides yields

$$w_i f_i\left(\lambda x + (1-\lambda)y\right) \leq \lambda w_i f_i(x) + (1-\lambda) w_i f_i(y), \quad \forall i \in \{1, \ldots, k\}$$

Taking summation on both sides for i from 1 to k will give

$$\begin{aligned} \sum_{i=1}^{k} w_i f_i\left(\lambda x + (1-\lambda)y\right) &\leq \sum_{i=1}^{k} \left(\lambda w_i f_i(x) + (1-\lambda) w_i f_i(y)\right) \\ &= \lambda \sum_{i=1}^{k} w_i f_i(x) + (1-\lambda) \sum_{i=1}^{k} w_i f_i(y) \\ f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

This proves $f(x) = \sum_{i=1}^{k} w_i f_i(x)$ is a convex function.

(b) Put $h := \max(f_1, f_2)$. Then, for any $x, y \in \mathbb{R}^N$ and $0 \leq \lambda \leq 1$, we have

$$h\left(\lambda x + (1-\lambda)y\right) = \max\left\{f_1\left(\lambda x + (1-\lambda)y\right), f_2\left(\lambda x + (1-\lambda)y\right)\right\}$$

Since $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are convex, so

$$\begin{cases} f_1\left((\lambda x + (1-\lambda)y\right) \leq \lambda f_1(x) + (1-\lambda) f_1(y) \\ f_2\left(\lambda x + (1-\lambda)y\right) \leq \lambda f_2(x) + (1-\lambda) f_2(y) \end{cases}$$

Taking maximum on both sides, we have

$$\begin{aligned} h(\lambda x+(1-\lambda)y) &= \max\left\{f_1\left(\lambda x+(1-\lambda)y\right), f_2\left(\lambda x+(1-\lambda)y\right)\right\} \\ &\leq \max\left\{\lambda f_1(x)+(1-\lambda)f_1(y), \lambda f_2(x)+(1-\lambda)f_2(y)\right\} \end{aligned}$$

and as $\lambda \in (0,1)$ is a constant thus

$$\begin{aligned} &\max\left\{\lambda f_1(x)+(1-\lambda)f_1(y), \lambda f_2(x)+(1-\lambda)f_2(y)\right\} \\ \leq\ &\lambda \max\left\{f_1(x), f_2(x)\right\}+(1-\lambda)\max\left\{f_1(y), f_2(y)\right\} \\ =\ &\lambda h(x)+(1-\lambda)h(y) \end{aligned}$$

This proves that $h := \max(f_1, f_2)$ is convex.

Q3. Yes, the implication is true. The proof is proceeded as follows:

Strong Convexity $\implies$ Strict Convexity:

Suppose f is strongly convex, then there exists a constant $\rho > 0$ such that for any $x, y \in \mathbb{R}^N$ and $\lambda \in (0,1)$, then

$$f(\lambda x+(1-\lambda)y)-\frac{\rho}{2}\|\lambda x+(1-\lambda)y\|^2 \leq \lambda f(x)-\frac{\lambda\rho}{2}\|x\|^2+(1-\lambda)f(y)-\frac{(1-\lambda)\rho}{2}\|y\|^2$$

Now, we discuss

$$\begin{aligned} &\frac{\rho}{2}\|\lambda x+(1-\lambda)y\|^2-\frac{\lambda\rho}{2}\|x\|^2-\frac{(1-\lambda)\rho}{2}\|y\|^2 \\ =\ &\frac{\rho}{2}\left[\lambda^2\|x\|^2+2\lambda(1-\lambda)\langle x,y\rangle+(1-\lambda)^2\|y\|^2-\lambda\|x\|^2-(1-\lambda)\|y\|^2\right] \\ =\ &\frac{\rho}{2}\left[\lambda(\lambda-1)\|x\|^2+2\lambda(1-\lambda)\langle x,y\rangle+\lambda(\lambda-1)\|y\|^2\right] \\ =\ &\frac{\rho\lambda(\lambda-1)}{2}\left[\|x\|^2+2\langle x,y\rangle+\|y\|^2\right] \\ <\ &\frac{\rho\lambda(\lambda-1)}{2}\left[\|x\|^2-2\|x\|\|y\|+\|y^2\|\right] \qquad (\because \lambda(\lambda-1)<0 \text{ and } \langle x,y\rangle \geq -\|x\|\|y\|) \\ =\ &\frac{\rho\lambda(\lambda-1)}{2}\left[\|x\|-\|y\|\right]^2 \\ <\ &0 \end{aligned}$$

Thus, this follows that

$$\begin{aligned} f\left(\lambda x+(1-\lambda)y\right) &\leq \lambda f(x)+(1-\lambda)f(y)+\frac{\rho}{2}\|\lambda x+(1-\lambda)y\|^2-\frac{\lambda\rho}{2}\|x\|^2-\frac{(1-\lambda)\rho}{2}\|y\|^2 \\ &< \lambda f(x)+(1-\lambda)f(y) \end{aligned}$$

so f is strictly convex.

Next, to prove that Strict Convexity $\implies$ Convexity:

Suppose f is strictly convex, then for any $x, y \in \mathbb{R}^N$ and $\lambda \in (0,1)$,

$$f\left(\lambda x+(1-\lambda)y\right) < \lambda f(x)+(1-\lambda)f(y)$$

and this is equivalent to

$$f\left(\lambda x+(1-\lambda)y\right) \leq \lambda f(x)+(1-\lambda)f(y)$$

and this completes the proof.